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Ortho and Causal Closure as a Closure Operations in the Causal Logic

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We investigate two different closure operations (ortho and causal closure) generated by a causal structure. In the case of orthogonal sets bounded in time two closure operations coincide and a lattice of double orthoclosed sets in this case is orthomodular.

KEY WORDS: quantum logic; orthogonality relation; orthomodularity.

1. INTRODUCTION

The quantum logic approach to quantum mechanics needs orthomodularity (Pták and Pulmannová, 1991). It was shown in (Cegła and Jadczyk, 1977; Mayet, 1995) that the orthomodular structure appears naturally in special relativity if one defines the orthogonality relation as the space-like or light-like separation in Minkowski space-time. The main result states that the family of double orthoclosed sets (double cones) in Minkowski space forms a complete orthomodular lattice (Cegła and Florek, 2005).

In the present paper we shall study an orthogonality space $(Z, \perp)$ where the orthogonality relation is generated by the distinguished family G of subsets covering a space Z. Two points x, y in the space Z are *orthogonal* $x \perp y$ if there is no $f \in G$ such that $\{x, y\} \subseteq f$.

In the first part of the paper we consider two operations $A \to A^{\perp}$ where $A^{\perp}$ is an *orthogonal complement* of A and $A \to D(A)$ where $D(A)$ is a *causal closure* of A (see Definition 2.1). It is shown that $D(A)$ is a closure operation for the complete lattice which is formed by the family of sets with the property

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$A = D(A)$. We also consider the relations between two closure operations $D(A)$ and $A^{\perp\perp}$.

In the second part we consider the family G given by the graphs of some functions covering the space $Z = \mathbb{R} \times X$ where $\mathbb{R}$ is the real line and X is an arbitrary topological space. We say that a set $S \subseteq \mathbb{R} \times X$ is *bounded in time* if there exists a strip $[t_1, t_2] \times X$ containing S.

We prove that if S is an orthogonal set bounded in time then $D(S) = S^{\perp\perp}$. We also prove that $\zeta_b(Z, \perp) := \{S^{\perp\perp} \subseteq Z : \ S \text{ is bounded in time}\}$ is an orthomodular lattice and the following equalities are satisfied $\zeta_b(Z, \perp\ ) = \{S^{\perp\perp} : S$ is an orthogonal set bounded in time$\} = \{D(S) : S$ is an orthogonal set bounded in time$\}$.

2. CAUSAL CLOSURE AND ORTHOCLOSURE GENERATED BY A CAUSAL STRUCTURE

By a *causal structure* of a set Z we mean a non-empty family G of sets covering the set Z. Every element f belonging to G is called a *causal path*. Let us denote by $\beta(z) := \{f \in G : z \in f\}$ the set of all paths containing z.

Definition 2.1. A point $z \in Z$ is *causally controlled* by a set A if

$$\underset{f \in \beta(z)}{\forall} f \cap A \neq \emptyset.$$

A causal closure of A is the set of all points causally controlled by A and is denoted by $D(A)$

$$D(A) := \{z \in Z : \underset{f \in \beta(z)}{\forall} f \cap A \neq \emptyset\}.$$

An orthogonal complement of A is the set of all points orthogonal to A and is denoted by $A^{\perp}$

$$A^{\perp} := \{z \in Z : \underset{f \in \beta(z)}{\forall} f \cap A = \emptyset\}.$$

It is easy to see the following implications:

$$f \cap D(A) \neq \emptyset \Longrightarrow f \cap A \neq \emptyset, \tag{2.1}$$

$$f \cap A^{\perp} \neq \emptyset \Longrightarrow f \cap A = \emptyset. \tag{2.2}$$

Lemma 2.1. *The map $D : 2^{Z} \to 2^{Z}$ has the following properties:*

(i) $A \subseteq D(A)$,
(ii) *if* $A \subseteq B$ *then* $D(A) \subseteq D(B)$,
(iii) $D(A) = D(D(A))$.

Proof: From Definition 2.1, *(i)* and *(ii)* are obvious. It is enough to prove that $D(D(A)) \subseteq D(A)$. By implication (2.1) we have:

$$z \in D(D(A)) \Leftrightarrow \underset{f \in \beta(z)}{\forall} f \cap D(A) \neq \emptyset \Rightarrow \underset{f \in \beta(z)}{\forall} f \cap A \neq \emptyset \Leftrightarrow z \in D(A). \quad \square$$

The family of causally closed sets $\zeta(Z, D) := \{A \subseteq Z \ : \ A = D(A)\}$ forms a complete lattice partially ordered by set-theoretical inclusion with l.u.b. and g.l.b. given respectively by

$$\bigvee A_i = D(\bigcup A_i), \qquad \bigwedge A_i = \bigcap A_i .$$

It is well known (Birkhoff, 1967) that the orthogonal map $\perp : 2^Z \to 2^Z$ has the following properties:

$$A \subseteq A^{\perp\perp} = (A^{\perp})^{\perp},$$
$$\text{if } A \subseteq B \text{ then } B^{\perp} \subseteq A^{\perp},$$
$$A \cap A^{\perp} = \emptyset,$$
$$A^{\perp} = A^{\perp\perp\perp},$$

from which follows that $\perp\perp$ is a closure operation (an *orthoclosure*).

The family of double orthoclosed sets $\zeta(Z, \perp) := \{A \subseteq Z : A = A^{\perp\perp}\}$ forms a complete ortholattice partially ordered by set theoretical inclusion with l.u.b. and g.l.b. given respectively by

$$\bigvee A_i = (\bigcup A_i)^{\perp\perp}, \qquad \bigwedge A_i = \bigcap A_i .$$

Of course there are relations between two closure operations D and $\perp\perp$.

Lemma 2.2. *The maps* $D : 2^Z \to 2^Z$ *and* $\perp : 2^Z \to 2^Z$ *have the following properties:*

(i) $[D(A)]^{\perp} = A^{\perp} = D(A^{\perp})$,
(ii) $D(A) \subseteq A^{\perp\perp}$.

Proof: (i) By Lemma 2.1 $A^{\perp} \subseteq D(A^{\perp})$ and $A \subseteq D(A)$ from which it follows that $[D(A)]^{\perp} \subseteq A^{\perp}$. Therefore it is enough to prove only $D(A^{\perp}) \subseteq [D(A)]^{\perp}$. By implication (2.1) and (2.2) we have:

$$z \in D(A^{\perp}) \Leftrightarrow \underset{f \in \beta(z)}{\forall} f \cap A^{\perp} \neq \emptyset \Rightarrow \underset{f \in \beta(z)}{\forall} f \cap D(A) = \emptyset \Leftrightarrow z \in [D(A)]^{\perp} .$$

(ii) Because $A \subseteq A^{\perp\perp}$ then $D(A) \subseteq D(A^{\perp\perp}) = A^{\perp\perp}$. $\square$

Corollary 2.1. *From the Lemma 2.1 and 2.2. we get the formula*

$$A \subseteq D(A) \subseteq A^{\perp\perp} .$$

The simplest figures illustrating the relations shall be given in the Example 1 in the Section 4.

3. THE EQUIVALENCE BETWEEN THE CAUSAL CLOSURE AND ORTHOCLOSURE

From now on we shall use the causal structure introduced in (Cegła and Florek, 2005).

Let $Z = \mathbb{R} \times X$ be a topological product of the real line $\mathbb{R}$ and arbitrary topological space X. We denote by p a canonical projection of $\mathbb{R} \times X$ on $\mathbb{R}$. Let G be a family of sets given by the graphs of continuous functions $f : \mathbb{R} \to X$.

For $a \in Z$ and $A \subseteq \mathbb{R}$ we define:

$$a^- := \left\{z \in Z : p(z) \leq p(a) \wedge \underset{g \in \beta(z)}{\exists} a \in g\right\},$$

$$a^+ := \left\{z \in Z : p(z) \geq p(a) \wedge \underset{g \in \beta(z)}{\exists} a \in g\right\},$$

$$A^- := \bigcup_{a \in A} a^-,$$

$$A^+ := \bigcup_{a \in A} a^+.$$

It is easy to see that $A^{\perp} = \{z \in Z : \underset{f \in \beta(z)}{\forall} f \cap A = \emptyset\} = (A^+ \cup A^-)'$, where the prime symbol ($'$) means the set complement in Z.

For $A \subseteq Z$ and $f \in G$ we denote

$$\langle f, A \rangle := p(f \cap A) = \{p(z) \in \mathbb{R} : \; z \in f \cap A\}.$$

Hence by the property of projection p we have

$$\langle f, A^{\perp} \rangle = (\langle f, A^+ \rangle \cup \langle f, A^- \rangle)',$$

where the prime symbol ($'$) means the set complement in $\mathbb{R}$. Now we shall introduce the restrictions for the family G. We assume that G satisfies the following conditions:

$$(*) \qquad \underset{x,y,z \in Z}{\forall} \quad (x \in y^+ \wedge \; y \in z^+ \Rightarrow x \in z^+),$$

$$(**) \qquad \underset{z \in Z}{\forall} \quad z^+ \setminus \{z\} \quad \text{and} \quad z^- \setminus \{z\} \quad \text{are open sets in } \mathbb{R} \times X.$$

A set $S \subseteq Z$ is an *orthogonal set* iff $\underset{x,y \in S}{\forall} x \neq y, x \perp y$.

It was shown in (Cegła and Florek, 2005) that the above conditions $(*)$ and $(**)$ give the following one:

$(***)$ every orthoclosed set $A = A^{\perp\perp}$
is generated by a maximal orthogonal set
$S \subseteq A$ as follows $S^{\perp\perp} = A^{\perp\perp}$,

and by (Foulis and Randall, 1971) we have

$(****)$ the set $\zeta(Z, \perp) = \{A \subseteq Z : A = A^{\perp\perp}\}$
is complete orthomodular lattice.

The following results were obtained in (Cegła and Florek, 2005). If $f \in G$, $A \subseteq Z$ and $f \cap A = \emptyset$ then:

(i) $\langle f, A^- \rangle = \emptyset$ or $\langle f, A^- \rangle = \mathbb{R}$ or $\langle f, A^- \rangle = (-\infty, s)$ where $s = \sup\langle f, A^- \rangle$,
(ii) $\langle f, A^+ \rangle = \emptyset$ or $\langle f, A^+ \rangle = \mathbb{R}$ or $\langle f, A^+ \rangle = (t, \infty)$ where $t = \inf\langle f, A^+ \rangle$,
(iii) $\langle f, A^\perp \rangle$ is closed and connected subset of $\mathbb{R}$.

Lemma 3.1. *If S is an orthogonal set bounded in time then*

$$f \cap S = \emptyset \Rightarrow f \cap S^\perp \neq \emptyset\,.$$

Proof: From the boundness in time we see immediately that

$$\langle f, S^+ \rangle \neq \mathbb{R} \text{ and } \langle f, S^- \rangle \neq \mathbb{R}\,.$$

Let us the first assume that $\langle f, S^+ \rangle \neq \emptyset$ and $\langle f, S^- \rangle \neq \emptyset$. Hence by (i) and (ii) $\langle f, S^- \rangle = (-\infty, s)$ and $\langle f, S^+ \rangle = (t, \infty)$. By the orthogonality of S

$$\langle f, S^+ \rangle \cap \langle f, S^- \rangle = \emptyset\,,$$

hence $s \leq t$ and

$$\langle f, S^\perp \rangle = (\langle f, S^+ \rangle \cup \langle f, S^- \rangle)' = [s, t] \neq \emptyset.$$

On the other hand if we assume that $\langle f, S^+ \rangle = \emptyset$ or $\langle f, S^- \rangle = \emptyset$ then also $\langle f, S^\perp \rangle = \langle f, S^- \rangle' \neq \emptyset$ or $\langle f, S^\perp \rangle = \langle f, S^+ \rangle' \neq \emptyset$. □

Theorem 3.1. *If S is an orthogonal set bounded in time then*

$$S^{\perp\perp} = D(S)\,.$$

Proof: It is enough to prove that $S^{\perp\perp} \subseteq D(S)$. If $z \in S^{\perp\perp}$ and $z \notin D(S)$ then exists $f \in \beta(z)$ such that $f \cap S = \emptyset$ and $f \cap S^\perp = \emptyset$. By Lemma 3.1. it is impossible. □

Lemma 3.2. *Let $P = [t_1, t_2] \times X$ be a strip and $B \subseteq P$.*
If S is the maximal orthogonal set in $P \cap B^{\perp\perp}$ then $S^{\perp\perp} = B^{\perp\perp}$.

Proof: Let $x \in B^{\perp\perp} \backslash (P \cap B^{\perp\perp})$. There are two cases:

1. $p(x) \geq t_2$,
2. $p(x) \leq t_1$.

We shall examine the case 1. Because $x \notin B^{\perp}$ then exists $g \in \beta(x)$ and $y \in g$ such that $y \in B$. Let $z \in g$ and $p(z) = t_2$ Because $p(y) \leq p(z) \leq p(x)$ and $p(y), p(x) \in \langle f, B^{\perp\perp} \rangle$ so, by (iii) $p(z) \in \langle f, B^{\perp\perp} \rangle$. Hence $z \in B^{\perp\perp}$. Because S is the maximal orthogonal set in $P \cap B^{\perp\perp}$ and $z \in P \cap B^{\perp\perp}$ we conclude that there exists $h \in \beta(z)$ such that $h \cap S \neq \emptyset$. Because $z \in S^{+}$ and $x \in z^{+}$ so by the transitivity condtition $(*)$ $x \in S^{+}$. So we proved that S is the maximal orthogonal set in $B^{\perp\perp}$. Using $(***)$ we have $S^{\perp\perp} = B^{\perp\perp}$. □

Using Theorem 3.1 and Lemma 3.2 we are able to prove our main result.

Theorem 3.2. *The set $\zeta_b(Z, \perp) := \{S^{\perp\perp} : S$ is bounded in time$\}$ is an orthomodular lattice and $\zeta_b(Z, \perp) = \{S^{\perp\perp} : S$ is an orthogonal set bounded in time$\} = \{D(S) : S$ is an orthogonal set bounded in time$\}$.*

Proof: From Lemma 3.2 and Theorem 3.1. we get the equality

$$\begin{aligned} \zeta_b(Z, \perp) &= \{S^{\perp\perp} : S \text{ is an orthogonal set bounded in time}\} \\ &= \{D(S) : S \text{ is an orthogonal set bounded in time}\}. \end{aligned}$$

Now we shall check the following properties:

(i) $\zeta_b(Z, \perp)$ is closed for the l.u.b.,
(ii) $\zeta_b(Z, \perp)$ is closed for the orthocomplementation,
(iii) $\zeta_b(Z, \perp)$ is closed for the g.l.b.

(i) Let S_1, S_2 are bounded in time sets such that $S_1^{\perp\perp} = A$, $S_2^{\perp\perp} = B$. It is enough to prove that $(S_1 \cup S_2)^{\perp\perp} = (A \cup B)^{\perp\perp}$. It is easy to see that

$$A = S_1^{\perp\perp} \subseteq (S_1 \cup S_2)^{\perp\perp},$$

$$B = S_2^{\perp\perp} \subseteq (S_1 \cup S_2)^{\perp\perp}.$$

From this we get $(A \cup B)^{\perp\perp} \subseteq (S_1 \cup S_2)^{\perp\perp}$. The contrary relation is obvious.

(ii) Let S be the orthogonal set bounded in time such that

$$S^{\perp\perp} = A.$$

Let $P = [t_1, t_2] \times X$ be a strip such that $S \subseteq P$.
It is enough to prove that

$$(P \cap A^{\perp})^{\perp\perp} = A^{\perp}.$$

At first we shall prove that $A^{\perp} \subseteq D(P \cap A^{\perp})$.

Let $x \in A^{\perp} \backslash (P \cap A^{\perp})$. There are two cases:

1. $p(x) \geq t_2$,
2. $p(x) \leq t_1$.

   We shall examine the case 1. Let $f \in \beta(x)$, $z \in f$ and $p(z) = t_2$. We shall see that $z \in A^{\perp}$. If $z \notin A^{\perp}$ then there exists $h \in \beta(z)$, $h \cap A \neq \emptyset$. But $D(S) = A$ (Theorem 3.1) then there exists $w \in h \cap S$, but $p(w) \leq p(z) \leq p(x)$ so $z \in w^{+}$ and $x \in z^{+}$. Hence by the transitivity condition $(*)$ $x \in w^{+}$. But $w \in S \subset A$ so $x \notin A^{\perp}$ which contradicts the assumption $x \in A^{\perp} \setminus (P \cap A^{\perp})$.

   Hence we see that $A^{\perp} \subseteq D(P \cap A^{\perp}) \subseteq (P \cap A^{\perp})^{\perp\perp}$.

   The contrary relation is obvious.

(iii) By the property of the orthogonality relation we have: $A \cap B \subseteq (A \cap B)^{\perp\perp} \subseteq (A^{\perp} \cup B^{\perp})^{\perp} \subseteq A^{\perp\perp} \cap B^{\perp\perp}$. It is enough to see that if $A = A^{\perp\perp}$, $B = B^{\perp\perp}$ then $A \cap B = (A^{\perp} \cup B^{\perp})^{\perp} = [(A^{\perp} \cup B^{\perp})^{\perp\perp}]^{\perp}$ and use (i) and (ii).

At the end observe that by (i), (ii), (iii) and condition $(* * **)$ the set $\{S^{\perp\perp} : S$ is bounded in time $\}$ is an orthomodular lattice. □

Remark 3.1. The set $\zeta_b(Z, \perp)$ is not σ-complete lattice (see Example 2).

If we consider the following family $\zeta_P(Z, \perp) := \{S^{\perp\perp} : S \subseteq P\}$ where $P = [t_1, t_2] \times X$ is a fix strip then by Theorem 3.2 $\zeta_P(Z, \perp)$ is complete orthomodular lattice.

4. THE EXAMPLES

We shall consider the space $Z = \mathbb{R}_t \times \mathbb{R}$, where $\mathbb{R}_t = \mathbb{R}$ (denotes the time) and the causal structure is given by a family of functions G satisfying sharp Lipschitz condition.

$$G = \{f : \mathbb{R}_t \to \mathbb{R};\ \ |f(t_1) - f(t_2)| < |t_1 - t_2|\}.$$

The family G satisfies the conditions $(*)$ and $(**)$ of Section 3. The following figures illustrate the examples in two- dimensional Minkowski space-time.

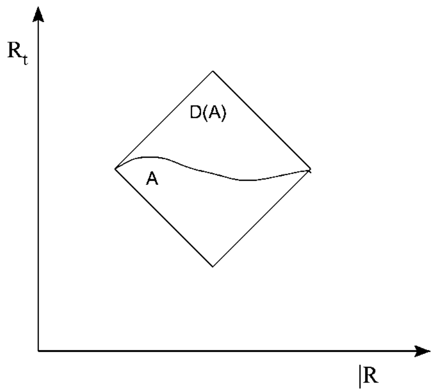

Fig. 1. $A \neq D(A) = A^{\perp\perp}$.

Example 4.1. In Fig. 1 the set A is an orthogonal set bounded in time and $D(A) = A^{\perp\perp}$. In Fig. 2 a set A as a hiperbola is an orthogonal set unbounded in time, and $D(A) \neq A^{\perp\perp} = Z$.

If we restrict our considerations to the set $A \subseteq \{(t, t);\ t \in [t_1, t_2]\}$ then A is orthogonol bounded in time and $A = D(A) = A^{\perp\perp}$.

Example 4.2. Let $A := \{(t, t);\ t \in W\}$ where W denotes the set of all rational numbers.

It is not difficult to see that $A = A^{\perp\perp}$ hence $A = \bigvee_{t \in W} \{(t, t)\}$. Of course $A \notin \zeta_b(Z, \perp)$ and $\{(t, t)\} \in \zeta_b(Z, \perp)$ for any $t \in \mathbb{R}$. So $\zeta_b(Z, \perp)$ is not σ-complete lattice.

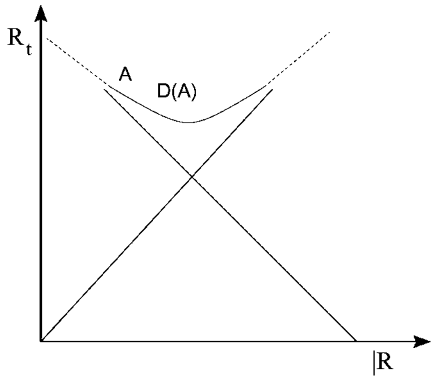

Fig. 2. $A \neq D(A) \neq A^{\perp\perp} = Z$.

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